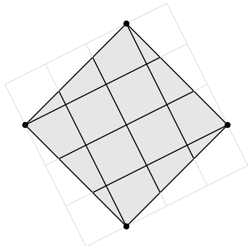
501. "If $f(x)$ and $g(x)$ are linearly related, then $f'(x)$ and $g'(x)$ are directly proportional."

Is this true or false?

502. The diagram below shows a grid of unit squares and a larger shaded region.



Find the area of the shaded region.

503. Three interior angles of a quadrilateral are given as $\frac{\pi}{5}$, $\frac{2\pi}{5}$ and $\frac{3\pi}{5}$ radians. Find the fourth angle.

504. Variables a, b, c, d are related by $a \propto b^2$, $b \propto c^3$, $c \propto d^4$. Find the relationship between a and d.

505. It is given that the graphs $x^2+y^2 = 1$ and $x+y = k$ have at least one point of intersection. Determine all possible values of the constant k, giving your answer in set notation.

506. Show that $(x+1)$ leaves a non-zero remainder when dividing $4x^3 - 12x^2 + 18$.

507. Three cards are put into a hat. One is red on both sides, one is green on both sides, one is red on one side and green on the other. Find the probability that, if two cards are drawn out and laid on the table, they both show red.

508. A bridge crossing a stream consists of a uniform wooden beam of mass 20 kg, laid symmetrically with 60% of its length resting on flat ground.

Modelling the beam in three sections, as denoted by the dotted lines, find the upward force exerted by each of the outer sections on the middle section.

509. For a function f and constants a, b in the domain of f, state whether the following is true or false:

$$f(x)\Big|_{x=b} - f(x)\Big|_{x=a} \equiv \Big[-f(x)\Big]_{x=b}^{x=a}$$

510. An odd integer can be expressed as $2k + 1$, where $k \in \mathbb{Z}$, i.e. where k is any integer.

(a) Show that $(2k + 1)^2 \equiv 2(2k^2 + 2k) + 1$.

(b) Hence, prove that the square of an odd number is always odd.

511. Show that, at half past three, the angle between the hands of a clock is $\frac{5\pi}{12}$ radians.

512. This question concerns proving that a triangle of side lengths $(a, b, c) = (5, 6, 7)$ will not fit inside a rectangle with a side of length 4.

(a) Use the cosine rule to show that the largest angle in the triangle is $C = \arccos \frac{1}{5}$.

(b) Determine the area of the triangle.

(c) Hence, find the length of the perpendicular from point C to the side of length c and thus prove the result.

513. Give the meaning of the following adjectives used in mechanical modelling:

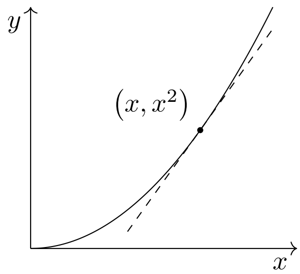
(a) "smooth",

(b) "rigid",

(c) "light" (as opposed to heavy).

514. Determine the four roots of $\sin 2\theta = \frac{\sqrt{3}}{2}$ in $[0, 2\pi)$.

515. The diagram shows $y = x^2$, with a dashed tangent drawn at a generic point (x, x^2):



To determine the gradient of the tangent shown above, the following limit has been constructed.

$$\lim_{\delta x \to 0} \frac{x^2 - (x - \delta x)^2}{\delta x}.$$

(a) The fraction inside the limit is the gradient of a chord. Sketch this chord on a copy of the diagram.

(b) Explain the meaning and role of δx.

(c) By expanding and simplifying the numerator, prove that the derivative of x^2 is $2x$.

516. A triangle has two sides whose lengths are 20 and 21. The third side has length $c \in \mathbb{N}$. The area of the triangle is 126. Determine the value of c.

517. True or false?

(a) $\dfrac{d}{dx}(x + 1) \equiv \dfrac{d}{dx}(x + 2)$,

(b) $\displaystyle\int (x + 1)\, dx \equiv \int (x + 2)\, dx$.

518. State, with a reason, whether getting cards of the same suit is more probable if the cards are picked
    (a) with replacement,
    (b) without replacement.

519. The unit circle is $x^2 + y^2 = 1$. Using geometric methods, or otherwise, prove that

$$\frac{dy}{dx} = -\frac{x}{y}.$$

520. Explain how Newton's first law may be thought of as a special case of Newton's second law.

521. By treating it as a quadratic, solve the equation

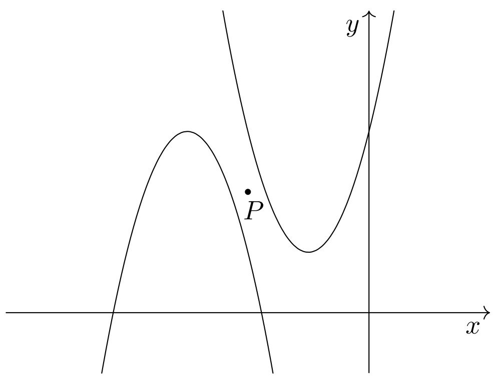
$$x^5 - x^2\sqrt{x} - 992 = 0.$$

522. Brahmagupta studied triangles with side lengths a, b, c generated by

$$\begin{aligned} a &= n(m^2 + k^2), \\ b &= m(n^2 + k^2), \\ c &= (m+n)(mn - k^2). \end{aligned}$$

Prove that the perimeter of such a triangle may be expressed as $P = 2mn(m + n)$.

523. The parabolae $y = x^2 + 4x + 6$ and $-x^2 - 12x - 30$ are shown on the diagram below. They may be transformed onto each other by a rotation of 180° around point P.



Determine the coordinates of point P.

524. Give the interior angles of the following regular polygons in radians, as fractions of π:
    (a) a square,
    (b) a hexagon,
    (c) a dodecagon.

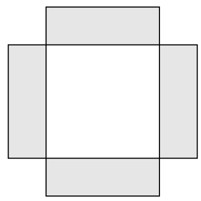
525. Prove that, if p_1 and p_2 are primes greater than 2, then $p_1p_2 + 1$ cannot be prime.

526. Find $\displaystyle\int_{-k}^{k} x^3 - x \, dx$ and interpret your result.

527. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ (if any) links statements ① and ② concerning a real number x:
    ① $x \in \{1\}$,
    ② $x \in \{1, 2\}$.

528. The equation $x^4 + y^4 = 400$ forms a closed loop. Determine whether the point $(3, 4)$ lies inside, on, or outside this loop.

529. In the diagram below, which is drawn to scale, two congruent rectangles are positioned at right angles to one another. Each has perimeter 32, and the shaded area is 48.



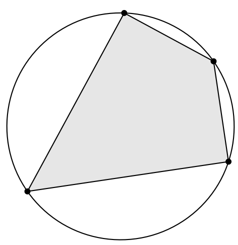
Determine the area of the central square.

530. A uniform beam is placed horizontally, supported at two points which divide the length of the beam in the ratio 2 : 3 : 4.
    (a) Draw a force diagram.
    (b) Hence, show that the ratio of the magnitude of the reaction forces at the supports is 1 : 5.

531. A wizard has velocity $\mathbf{i} + a\mathbf{j} + 2\mathbf{k}$ ms^{-1} and speed 3 ms^{-1}. Find all possible values of a.

532. At the point with x coordinate p, the tangent line to $y = x^2$ has equation $y = 2px + c$.
    (a) Explain the presence of the coefficient $2p$.
    (b) Show that a general tangent line to the curve $y = x^2$ has equation $y = 2px - p^2$.

533. The diagram shows a cyclic kite:



Prove that one of the kite's diagonals must be a diameter of the circumscribing circle.

534. In a game, three coins are tossed, then two, then one. Find the probability that a total of four tails show during the game.

535. Variables x and y are related by $y = x^3$. Sketch, on a single set of axes, the regions whose areas are calculated by the following integrals:
   (a) $\int_0^k y\,dx$,
   (b) $\int_0^{k^3} x\,dy$.

536. Prove that the product of six consecutive integers must end in the digit zero.

537. A student is trying to prove that $\text{f}(x) \equiv \text{g}(x)$, for some two algebraic functions f and g. He ends up, via correct algebraic reasoning, producing the following implication:

$$\text{f}(x) \equiv \text{g}(x) \implies 0 = 0 \;\checkmark$$

He claims that this proves the result. Explain the error in his logic.

538. The curve $y = x^{-\frac{1}{2}}$ has a normal drawn to it at the point with x coordinate 4. Show that the equation of this normal is $2y = 32x - 127$.

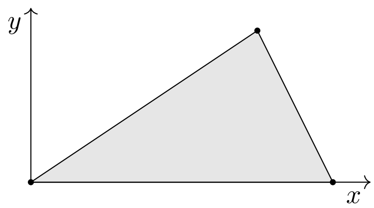
539. A set of data, whose mean is 4, is given as follows:

x	1	2	3	4	5
f	1	4	5	17	n

Find n.

540. By sketching or considering signs of factors, solve the inequality $(x-1)(x-2)(x-3) \leq 0$, giving your answer in set notation.

541. A triangle T has vertices $(0, 0)$, $(a, 0)$, and (b, c), where $a, b, c > 0$.



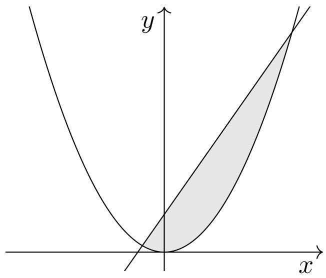
Show that the area of T does not depend on b.

542. Explain how you know that the following equation has no real roots:

$$\frac{(x^2 + a^2 + 1)(x^2 + b^2 + 1)}{(x^2 + c^2 + 1)(x^2 + d^2 + 1)} = 0.$$

543. A geometric sequence has nth term u_n. Show that $w_n = u_{n-1} + u_n$ is also geometric.

544. The curve $y = x^2$ and the line $y = 2x + 1$ enclose a region of the (x, y) plane.



   (a) Show that the intersections are at $x_1 = 1 - \sqrt{2}$ and $x_2 = 1 + \sqrt{2}$.
   (b) The area of the shaded region is calculated with the definite integral

$$A = \int_{x_1}^{x_2} 2x + 1 - x^2 \, dx.$$

   Explain the form of the integrand.
   (c) Show that $A = \frac{8}{3}\sqrt{2}$.

545. Find the term independent of x in the binomial expansion of the following expression:

$$\left(x - \frac{1}{x}\right)^4.$$

546. State, with a reason, whether $y = x^2$ intersects the following lines/curves:
   (a) $y = x + 1$,
   (b) $y = x^2 + 1$,
   (c) $y = x^3 + 1$.

547. State, with a reason, which of the following shapes has the larger area:
   ① a regular n-gon of side length l,
   ② a regular $(n + 1)$-gon of side length l.

548. Describe the locus of the following equation, for non-zero constants a, b:

$$x^2 + y^2 = (x - a)^2 + (y - b)^2.$$

549. Two inequalities are given below:

$$x^4 + y^4 < 1,$$
$$x + y > 2.$$

   (a) Sketch the region $x + y > 2$ on a set of axes.
   (b) Show that, for all points in the region, at least one of x or y is greater or equal to 1.
   (c) Hence, show that no (x, y) points satisfy both inequalities simultaneously.

550. A biquadratic in x is a quadratic in x^2. Solve the biquadratic $5x^4 - 6x^2 + 1 = 0$.

551. Let $X \sim N(0, 1)$. State, with a reason, whether the following variables are normally distributed:

   (a) $-X$,
   (b) $|X|$,
   (c) $10 - X$,
   (d) X^2.

552. True or false?

   (a) $\sin x = \sin y \implies x = y$,
   (b) $\sin x = \sin y \impliedby x = y$,
   (c) $\sin x = \sin y \iff x = y$.

553. Sketch the graph $x^2 + y^2 = x + y$.

554. Simplify the following expression, in which $a \neq 0$:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

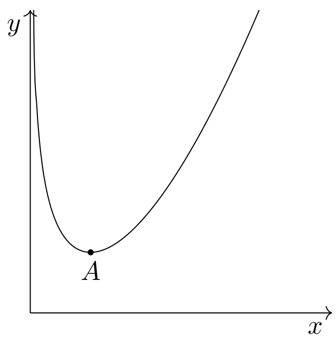
555. *Euclid's formula*, in which $p, q \in \mathbb{N}$ with $p > q$, gives triples (a, b, c) as follows:

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2.$$

   (a) Show that this generates Pythagorean triples.
   (b) Hence, prove that there are infinitely many right-angled triangles with integer side lengths.

556. Evaluate $1 + 2 + 4 + 8 + ... + 1048576$.

557. A curve is given by $y = \dfrac{(x-1)^2}{\sqrt{x}} + 1$.



   (a) Find $\dfrac{dy}{dx}$.
   (b) Hence, determine the coordinates of point A, at which the curve is stationary.

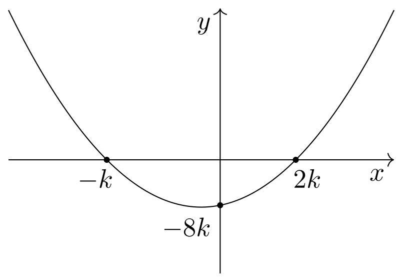
558. Give, in radians, the average of the interior angles of a decagon.

559. The mean of a and c is b. Prove that a, b, c are in arithmetic progression.

560. (a) Show that $\displaystyle\int_0^1 4x^3 - 5x^4 \, dx = 0$.

   (b) Without further calculation, explain how you know that the expression $4x^3 - 5x^4$ has a root in the interval $[0, 1]$.

561. Find the equation of the monic parabola shown, on which the axes intercepts have been marked, giving your answer in expanded polynomial form.



562. Find a, b, c such that the following identity holds:

$$20x^2 - 23x - 21 \equiv (ax + b)(ax - x + c).$$

563. The graph $y = x^2 + x$ is translated by the vector $2\mathbf{i} + 3\mathbf{j}$. Find, in the form $y = ax^2 + bx + c$, the equation of the new graph.

564. True or false?

   (a) $\mathrm{f}(a) = 0 \iff \big(\mathrm{f}(a)\big)^2 = 0$,
   (b) $\mathrm{f}(a) = 1 \iff \big(\mathrm{f}(a)\big)^2 = 1$.

565. A fridge of mass 75 kg is standing in a lift, which is accelerating downwards at 3 ms^{-2}. Find the force exerted by the fridge on the lift floor.

566. Prove the area formula $A = \frac{1}{2}ab \sin C$.

567. Give the range of each of the following functions, when each has the largest possible real domain:

   (a) $x \mapsto \sin 2x$,
   (b) $x \mapsto \cos 2x$,
   (c) $x \mapsto \tan 2x$.

568. Four points are $(0, 0)$, $(20, 0)$, $(0, 10)$, $(4, 8)$.

   (a) Show that three of these points are collinear.
   (b) Show that any other set of three points forms a right-angled triangle.

569. In this question, the notation $\dbinom{n}{r} \equiv {}^n\mathrm{C}_r$ is used.

Dixon's identity is

$$\sum_{k=-a}^{a} (-1)^k \binom{2a}{k+a}^3 \equiv \frac{(3a)!}{(a!)^3}.$$

Verify the identity for $a = 1$.

570. Most of the time, a human being has a mass on which weight and a reaction force act. State, with a reason, whether there is any physical scenario, as described by the Newtonian model, in which each of these quantities can drop to zero for a human being:

(a) mass,

(b) weight,

(c) reaction force.

571. Prove that, for all positive real numbers x, y,

$$x + y > 1 \implies 5x + 7y > 4.$$

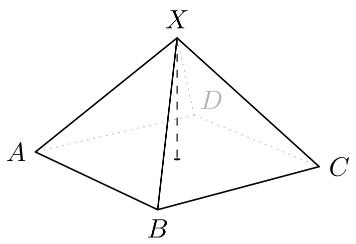
572. The following equations define two line segments. These line segments are chords of one circle.

$$x = s, \quad y = 2 - s, \quad s \in [0, 1]$$
$$x = -1 + t, \quad y = 1 + t, \quad t \in [0, 1]$$

Find the equation of the circle.

573. A sequence is defined iteratively by the following rule: the value of A_{n+1} is three greater than twice the value of A_n. The sequence is increasing. Show that it is neither arithmetic nor geometric.

574. The square-based pyramid shown below is formed of eight edges of unit length.



Determine the length of the dashed perpendicular shown dropped from X to $ABCD$.

575. "The curves $y = x^2 + 1$ and $y = -4x - 3x^2$ are tangent to one another." True or false?

576. Functions F and G are periodic, with periods 3 and 5 respectively. State the period of each of the following functions:

(a) $x \mapsto \mathrm{F}\left(\frac{1}{2}x + 1\right)$,

(b) $x \mapsto 3\,\mathrm{G}(x) + 2$,

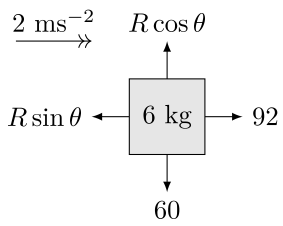
(c) $x \mapsto \mathrm{F}(x) + \mathrm{G}(x)$,

577. Find the area enclosed by the curve $y = x^2$ and the line $y = 4$.

578. Three coins are tossed simultaneously. Event S is defined as all three coins showing the same result, and S' is the complement of this event. Show that $\mathbb{P}(S') = 3\,\mathbb{P}(S)$.

579. Sketch the graph $x^3y^3 = 1$.

580. Four force components, whose lines of action are perpendicular, cause a 6 kg mass to accelerate as shown. The magnitudes of the force components are given in Newtons.



(a) Find the values of $R\sin\theta$ and $R\cos\theta$.

(b) Hence, find R and θ.

581. A quadrilateral Q has vertices at $(0, 0)$ and $(6, 0)$, and its diagonals intersect at $(3, 4)$. Find the set of possible values of the perimeter of Q.

582. Without multiplying out, solve the equation

$$(x - 1)(x - 2) + (x - 1)(x - 3) = 0.$$

583. The equation $x = 4y^2 + 12y + 11$ gives a parabola.

(a) Complete the square for y.

(b) Hence, state the coordinates of the vertex.

(c) Sketch the parabola.

584. A straight line is given parametrically as $x = t + 4$, $y = 3t$, for $t \in \mathbb{R}$. This line is then translated by the vector $5\mathbf{i}$. Write down the equation of the new line, in the same form.

585. Simplify $\dfrac{p^{\frac{5}{2}} - p^{\frac{3}{2}}}{p^{\frac{3}{2}} - p^{\frac{1}{2}}}$.

586. A circular ripple is spreading across a pond. Its radius in increasing linearly with time, at a rate of 5 centimetres per second. Find the rate of change of the circumference.

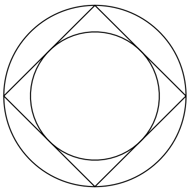
587. Consider the quartic $y = (x - p)^2(x - q)^2$, where p and q are constants satisfying $0 < p < q$.

(a) Without doing any calculations, explain how you know that the quartic has two stationary points on the x axis.

(b) Sketch the curve.

(c) Without using any calculus, write down, in terms of p and q, the x coordinate of the third stationary point on the curve.

588. A circle is inscribed in a square, which is inscribed in a circle.



Show that the circles have areas in the ratio 1 : 2.

589. Show that $\mathbf{r} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ is a unit vector.

590. A student suggests that the following is an identity for some suitable choice of constants P, Q:

$$\frac{1}{x} \equiv \frac{P}{1-x} + \frac{Q}{1+x}.$$

By multiplying up and comparing the coefficients of powers of x, or otherwise, prove that the student is incorrect.

591. Find simplified expressions for the sets

(a) $\mathbb{Z} \cup \mathbb{R}$,

(b) $\mathbb{Z} \cap \mathbb{N}$,

(c) $\mathbb{Z} \cup \mathbb{Q}$.

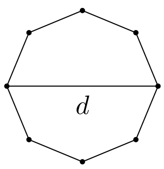
592. The derivative of the function $\text{f}(x) = x^2(1+x^n)$ is $\text{f}'(x) = x\big(a + bx^3\big)$, where $n, a, b \in \mathbb{N}$. Find n, a, b.

593. A *harmonic progression* is defined as a sequence of the reciprocals of an arithmetic progression. Prove that every harmonic progression is either constant or tends to zero.

594. Sketch a linear graph $y = \text{f}(x)$ for which

$$\int_0^1 \text{f}(x)\,dx = -2, \quad \text{f}(1) = 0.$$

595. The diagram shows a regular octagon of side length 1, with diameter d from vertex to vertex. The sine of $\frac{\pi}{8}$ radians is also given.



$$\sin\tfrac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

Find the exact value of d.

596. Prove that the product of two square numbers is a square number.

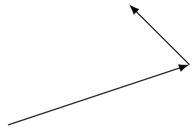
597. Solve $2^x \times 4^{x-1} \times 8^{x-2} = 1$.

598. You are given that g is a monic quadratic function, and that $\text{g}(2) = \text{g}'(2) = 0$. Sketch the following graphs, labelling all axis intercepts:

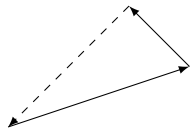
(a) $y = \text{g}(x)$,

(b) $x = \text{g}(y)$.

599. A particle is in equilibrium under the action of three forces. Vectors representing two of the three forces have been drawn tip-to-tail:



A vector representing the third force is to be added in the same fashion. Explain why this third force must produce a closed "triangle of forces".



600. Convert each of the following angular speeds to units of radians per second. Give your answers in standard form to 3sf.

(a) 45° per hour,

(b) 1.2×10^{-4} revolutions per minute.

——— End of 6th Hundred ———